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Threefolds whose canonical bundles are not numerically effective

by

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In this note, we announce an application of the previous paper [3] with some examples. The proof will be published elsewhere.

§1. Announcement.

We assume that k is an algebraically closed field of characteristic 0 and X is a non-singular projective 3-fold over k whose canonical bundle K_X is not numerically effective. We use the terminology of [3]. By Corollary & [3], X has an extremal ray R, which we fix in this section.

<u>Theorem</u> 1. There exists a morphism $\phi : X \longrightarrow Y$ to a projective variety Y such that (1) $\phi_*\mathcal{O}_X = \mathcal{O}_Y$, and (2) for any irreducible curve C in X, $[C] \in R$ if and only if $\dim \phi(C) = 0$. Furthermore, such a ϕ is unique up to an isomorphism.

The structure of this ϕ is given by the following theorems.

<u>Theorem</u> 2. The extremal ray R is not numerically effective if and only if $\dim Y = 3$. If these conditions are satisfied, then there exists an irreducible divisor D of X such that X is the blowing-up of Y by the ideal defining $\phi(D)$ (given the reduced structure), and we have either

(1) $\phi(D)$ is a non-singular curve and Y is non-singular; $\phi|_D : D \longrightarrow \phi(D)$ is a $\mathbb{P}^1$-bundle and $(D.\phi^{-1}(\eta)) = -1$ for any $\eta \in \phi(D)$,

(2) $Q = \phi(D)$ is a point and Y is non-singular; $D \cong \mathbb{P}^2$

and $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-1)$,

(3) $Q = \phi(D)$ is an ordinary double point of Y; $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_D(D) \cong p_1{}^* \mathcal{O}_{\mathbb{P}}(-1) \otimes p_2{}^* \mathcal{O}_{\mathbb{P}}(-1)$, where p_i is the i-th projection,

(4) $Q = \phi(D)$ is a double point of Y; D = an irreducible reduced singular quadric surface S in $\mathbb{P}^3$, $\mathcal{O}_D(D) \cong \mathcal{O}_S \otimes \mathcal{O}_{\mathbb{P}}(-1)$, or

(5) $Q = \phi(D)$ is a quadruple point of Y; $D \cong \mathbb{P}^2$, $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-2)$.

Let $\mathcal{O}_{Y,Q}$ be the local ring of Y at Q for cases (3), (4), and (5) in Theorem 2. Then we have

<u>Theorem</u> 3. (1) The divisor class group of $\mathcal{O}_{Y,Q}$ is 0 in cases (3) and (4), and $\mathbb{Z}/2\mathbb{Z}$ in case (5), and

(2) the completion $\mathcal{O}_{Y,Q}\hat{}$ of $\mathcal{O}_{Y,Q}$ is given by

$$\mathcal{O}_{Y,Q}\hat{} \cong \begin{cases} k[[x,y,z,u]]/(x^2 + y^2 + z^2 + u^2) & \text{case (3)}, \\ k[[x,y,z,u]]/(x^2 + y^2 + z^2 + u^3) & \text{case (4)}, \\ k[[x,y,z]]^{(2)} & \text{case (5)}, \end{cases}$$

where $k[[x,y,z]]^{(2)}$ is the invariant subring of $k[[x,y,z]]$ under the action of the involution $(x,y,z) \longmapsto (-x,-y,-z)$.

The remaining cases are treated by

<u>Theorem</u> 4. If R is numerically effective, then Y is non-singular, $\rho(X) = \rho(Y) + 1$, and we have either

(1) $\dim Y = 2$, and for an arbitrary geometric point η of Y, the scheme-theoretic fiber X_η is isomorphic to a conic of $\mathbb{P}^2{}_{k(\eta)}$, where $k(\eta)$ is the field of η (i.e. X_η is isomorphic to either a smooth conic, a reducible conic, or a double line,)

(2) $\dim Y = 1$, and for an arbitrary geometric point η of Y, X_η is an irreducible reduced surface such that $\omega_{X_\eta}^{-1}$ is ample, or

(3) $\dim Y = 0$, and X is a Fano 3-fold,(these 3-folds are classified by Iskovski [2].)

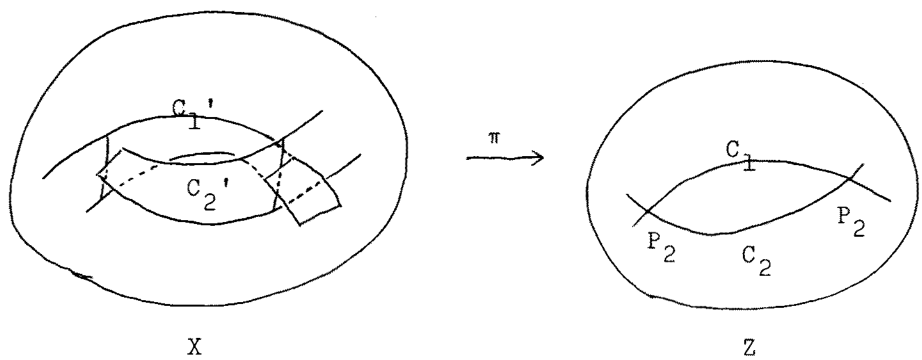
§2. Exceptional divisors.

The most interesting part of section 1 is Theorem 2. Examples for Theorem 2 can be given by considering birational morphisms.

Theorem 5. Let $\pi : X \longrightarrow Z$ be a birational morphism (which is not an isomorphism) of non-singular projective 3-folds. Then X contains an extremal rational curve ℓ such that (1) $\dim \pi(\ell) = 0$ and (2) ℓ is not numerically effective. Hence the exceptional set of π contains a divisor described in Theorem 2.

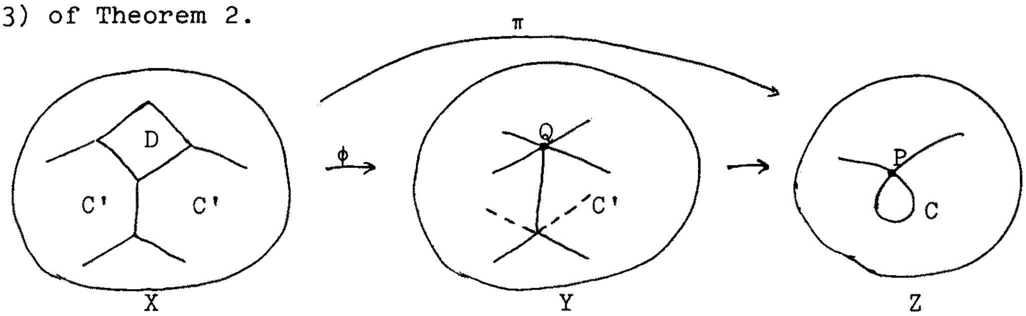
Examples 6. Let Z be a non-singular projective 3-fold.

(1): Let C_1 and C_2 be non-singular projective curves in Z intersecting transversally at 2 points P_1 and P_2. If we operate Hironaka's twisted blowing-up to C_1 and C_2 (e.g. blowing up C_1 first near P_1 and C_2 first near P_2), then the "blowing up" $\pi : X \longrightarrow Z$ does not have a divisor described in Theorem 2.



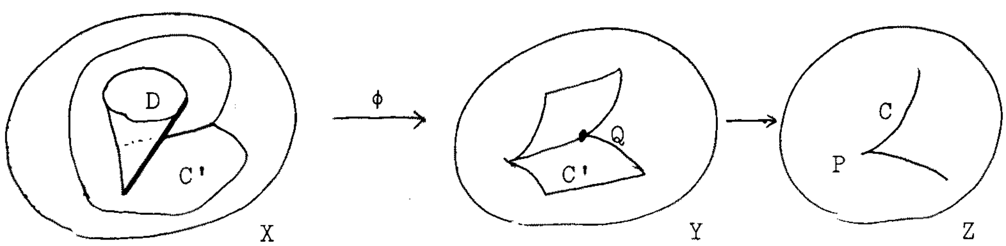
However, this does not contradict our theorems, because our X is not projective.

(2): Let C be an irreducible projective curve in Z with one ordinary double point P as singularities. If we blow up C, then the blown-up variety Y has one ordinary double point Q lying over P as singularities. If we resolve the singularity by blowing up Q and get a smooth 3-fold X, $\pi : X \longrightarrow Z$ and $\phi : X \longrightarrow Y$, then $D = \phi^{-1}(Q)$ is the divisor described in case (3) of Theorem 2.

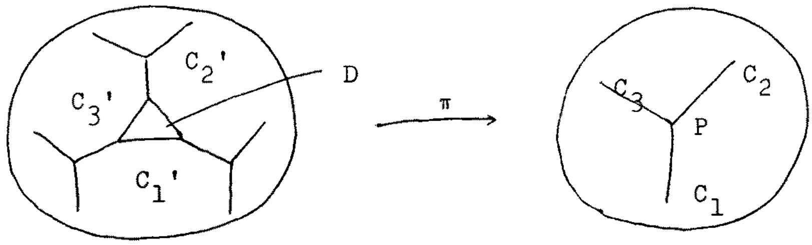


We remark that we can not start with an arbitrary ordinary double point because of Theorem 3, (1).

(3): Let C be an irreducible projective curve in Z with one ordinary cusp P as singularities. If we blow up C, the blown-up variety Y has one double point Q lying over P as singularities which falls in case (4) of Theorem 2. If we blow up Q to get a smooth 3-fold X, $\phi : X \longrightarrow Y$ and $\pi : X \longrightarrow Z$, then $D = \phi^{-1}(Q)$ is the divisor described in case (4).



(4): Let C_1, C_2, and C_3 be non-singular projective curves in Z intersecting transversally at one point P. If we operate Hironaka's modification in [1], we get a smooth projective 3-fold X and $\pi : X \longrightarrow Z$, and D, the divisorial part of $\pi^{-1}(P)$, is the divisor given in case 5 of Theorem 2.



We will finish this section by proving Theorem 5. The proof consists of a few easy lemmas. We keep the notation of Theorem 5 till the end of this section.

<u>Lemma</u> 7. $\pi_* : N(X) \longrightarrow N(Z)$ has the property

$$\pi_*\overline{NE}(X) \subseteq \overline{NE}(Z).$$

Indeed, we have $\pi_* NE(X) \subseteq NE(Z)$ by the definition of π_*, which implies Lemma 7 by continuity of π_*.

<u>Lemma</u> 8. There is an effective 1-cycle C on X such that $\pi_* C = 0$ and $(C.c_1(X)) > 0$.

<u>Proof</u>. Let E be the effective divisor on X such that Supp E is the exceptional set of π and $K_X = \pi^* K_Z + E$. We treat two cases.

Case 1: $\dim \pi(\text{Supp } E) = 1$.

By Bertini's theorem, there is a smooth hyperplanesection L

of Z such that $\pi^{-1}(L)$ is irreducible and non-singular. Then we have

$$\begin{aligned}(K_X.E.\pi^*L) &= (\pi^*K_Z.E.\pi^*L) + (E^2.\pi^*L)\\ &= (E.\pi^*(K_Z.L)) + (\mathcal{O}_{\pi^{-1}(L)}(E)^2)_{\pi^{-1}(L)}\\ &= (\pi_*E.K_Z.L) + (\mathcal{O}_{\pi^{-1}(L)}(E)^2)_{\pi^{-1}(L)}.\end{aligned}$$

Since $\pi_*E = 0$, we have $(\pi_*E.K_Z.L) = 0$. We have $(\mathcal{O}_{\pi^{-1}(L)}(E)^2)_{\pi^{-1}(L)} < 0$ because $C = E.\pi^*L$ ($\neq 0$, since $\dim \pi(\mathrm{Supp}\ E) = 1$) is an exceptional divisor of $\pi^{-1}(L) \longrightarrow L$. Hence $\pi_*C = 0$ and $(K_X.C) < 0$.

Case 2: $\dim \pi(\mathrm{Supp}\ E) = 0$.

Let M be a smooth hyperplanesection of X, hence M and E intersect properly and $M.E \neq 0$. Then we have

$$\begin{aligned}(K_X.E.M) &= (\pi^*K_Z.E.M) + (E^2.M)\\ &=(K_Z.\pi_*(E.M)) + (\mathcal{O}_M(E)^2)_M.\end{aligned}$$

Now $\pi_*(E.M) = 0$ because $\dim \pi(\mathrm{Supp}\ E) = 0$, and $(\mathcal{O}_M(E)^2)_M < 0$ because $E.M$ is an exceptional divisor of $M \longrightarrow \pi(M)$. Hence $C = E.M$ has the required property. q.e.d.

Lemma 9. There is an extremal rational curve ℓ on X such that $\pi_*\ell = 0$.

Proof. Let H be an arbitrary ample divisor on X and ε a small enough positive number so that $[C]$ given in Lemma 8 does not belong to $\overline{NE}_\varepsilon(X, H)$. By Theorem 3 in [3], $[C]$ is written as

$$[C] = \sum_{i=1}^{r} a_i\,[\ell_i] + V,$$

where $a_i \geq 0$, ℓ_i are extremal rational curves for all i, and $V \in \overline{NE}_\varepsilon(X)$. Hence $\Sigma a_i \pi_*[\ell_i] + \pi_* V = 0$ and $\pi_*[\ell_i], \pi_* V \in \overline{NE}(Z)$ by Lemma 7. Since Z is projective, we have $a_i \pi_*[\ell_i] = 0$ for all i and $\pi_* V = 0$ by Kleiman's criterion of projectivity: $\overline{NE}(Z) \cap \{-\overline{NE}(Z)\} = \{0\}$. Since $[C] \notin \overline{NE}_\varepsilon(X)$, there is at least one j such that $a_j \neq 0$. Then ℓ_j has the required property.

q.e.d.

Lemma 10. The curve ℓ in Lemma 9 is not numerically effective.

If E is the effective divisor on X given in the proof of Lemma 8, then

$$\begin{aligned}(\ell . E) &= (\ell . K_X) - (\ell . \pi^* K_Z) \\ &= (\ell . K_X) - (\pi_* \ell . K_Z) < 0.\end{aligned}$$

Thus Theorem 5 is proved, and it is easy to check the assertions in Examples 6.

References

[1] H. Hironaka, An example of non-Kaehlerian complex-analytic deformation of Kaehlerian complex structures, Ann. Math. Vol. 75 (1962), 190-208.

[2] V. A. Iskovskih, Fano 3-folds II, Math. USSR Izv. 11 (1977).

[3] S. Mori, The cone of effective 1-cycles, in the same volume.